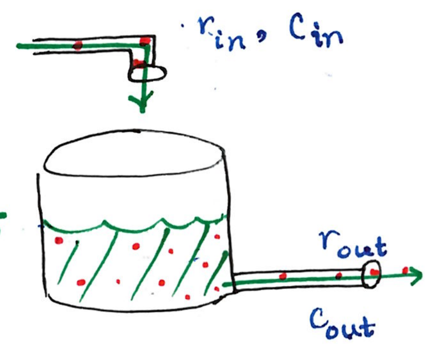
Sec 1.5 1^{st} order linear eqs

1^{st} order: 1^{st} derivative is highest appearing

linear: dependent/output variable y, y' are okay, but no $y^2, y^3, \ldots$ $(y')^2, \ldots$, nor yy'

- Ex: $\frac{dP}{dt} = P(3-P)$ ✗ (not linear b/c $3P - P^2$)
  - $F = mx''$ ✗ (linear, but not 1^{st} order)
  - $A' + e^t A = \cos(t)$ ✓ $(A = A(t))\left(A' = \frac{dA}{dt}\right)$
  - $A' + t^3 A = t^7$ ✓
  - $A' + t^3 A^2 = t^7$ ✗



Motivation: Mixture Problems

Goal: Model the amount $\underline{A(t)}$ of substance in solution (kgs, lbs, …) over time

- $V(t)$ = volume of solution (gal, L, …)
- r_{in}, r_{out} = flow rates $\left(\frac{gal}{min}, \frac{L}{hr}, \ldots\right)$
- c_{in}, c_{out} = concentration/density $\left(\frac{lbs}{gal}, \frac{kg}{L}, \ldots\right)$

- r_{in}, r_{out}, c_{in} usually constant, but $c_{out} = c_{out}(t)$ varies.

During time Δt, amount changes.

$$\Delta A = (A_{in} - A_{out})$$

$$\Delta A = \underbrace{(r_{in}c_{in} - r_{out}c_{out})}_{lbs/min \ldots} \Delta t.$$

So $\dfrac{\Delta A}{\Delta t} = r_{in}c_{in} - r_{out}\, c_{out}(t)$, so then with $\lim\limits_{\Delta t \to 0}$

$$\longrightarrow \quad \frac{dA}{dt} = r_{in}c_{in} - r_{out}\, c_{out}(t)$$

★ For simplicity, if solution is "well mixed", we can assume that $c_{out}(t) = \dfrac{A(t)}{V(t)}$, so our ODE is

$$\boxed{\frac{dA}{dt} = r_{in}\, c_{in} - \frac{r_{out}}{V(t)} A(t)}$$

or

$$\boxed{\frac{dA}{dt} + \frac{r_{out}}{V(t)} A(t) = r_{in}\, c_{in}} \quad \big(A' + P(t)A = Q(t) \big)$$

$$y' + P(x)y = Q(x)$$

General 1^{st} order linear eq: $\alpha(x)y' + \beta(x)y = \gamma(x)$

- "normalize" by dividing by $\alpha(x)$ to get into standard form:

$$y' + P(x)y = Q(x).$$

- If $P(x) = 0 \longrightarrow y' = Q(x)$ (separable) $\quad \dfrac{dy}{dx} = Q(x)$ $\to dy = Q(x)dx$
- If $Q(x) = 0 \longrightarrow y' = -P(x)y$ (separable)

How to solve eq from standard form:

$$y' + P(x)y = Q(x).$$

Idea: "complement" both sides with an integrating factor $\rho(x) \longrightarrow \rho y' + \rho P y = \rho Q$

where ρ is picked to make $\rho P = \rho'$

Why? Because if $\boxed{\rho' = \rho P,}$ then

$$\rho y' + \rho P y = \rho Q$$

$$\rightarrow \rho y' + \rho' y = \rho Q$$

$$\rightarrow (\rho y)' = \rho Q$$

and we can solve for y then!

$$\rightarrow \int (\rho y)' dx = \int \rho Q \, dx$$

$$\rho \cdot y + C = \int \rho Q \, dx$$

$$\boxed{y = \frac{\int \rho \cdot Q \, dx + C}{\rho}}$$

How do we find $\rho(x)$? We just need <u>any</u> ($\neq 0$) solution of $\rho' = \rho P$. (separable)

$$\frac{d\rho}{dx} = \rho P$$

$$\int \frac{d\rho}{\rho} = \int P \, dx$$

$$|\rho| = C e^{\int P dx}$$

$$\rho = \underbrace{\boxed{\pm C}}_{\text{"}C\text{"}} e^{\int P dx}$$

so let's <u>pick</u> $\boxed{\rho = 1 e^{\int P dx}}$

Read examples 1 and 2 from posted notes.

Example 1: Find a general solution for $\{y' - y = 3e^{-x}\}$

Eq is 1^{st} order linear, and is already in standard form:

$$y' - y = 3e^{-x} \longleftrightarrow y' + Py = Q \left(\begin{matrix} P = (-1) \\ Q = 3e^{-x} \end{matrix} \right)$$

Step 1: "complement" eq with $\rho = e^{\int P\,dx} = e^{\int (-1)\,dx} = e^{-x+C}$ *

* Because we are just picking any ρ that works, pick $C = 0$ so $\boxed{\rho = e^{-x}}$

("Template")

$$y' - y = 3e^{-x} \longleftrightarrow y' + Py = Q$$

$$e^{-x}y' \underbrace{- e^{-x}}y = 3e^{-2x} \qquad \rho y' + \rho P y = \rho Q$$

$$e^{-x}y' + \overbrace{(e^{-x})'}y = 3e^{-2x} \qquad \rho y' + \rho' y = \rho Q$$

Step 2: "integrate"

$$(e^{-x} \cdot y)' = 3e^{-2x} \qquad (\rho y)' = \rho Q$$

$$\int (e^{-x} \cdot y)'\,dx = \int 3e^{-2x}\,dx \qquad \int (\rho y)'\,dx = \int \rho Q\,dx$$

$$e^{-x}y = -\frac{3}{2}e^{-2x} + C \qquad \rho y = \int \rho Q\,dx + C$$

Step 3: "de-complement" (solve for y)

$$\underline{y = -\frac{3}{2}e^{-x} + Ce^{x}} \qquad y = \frac{\int \rho Q\,dx + C}{\rho}$$

Example 2: Find a general solution for $\{xy' - 5y = x^6\cos(x)\}$

Eq is 1^{st} order linear, but not in standard form:

change to $y' - \frac{5}{x}y = x^5 \cos x \longrightarrow y' + Py = Q$, ✓

$P = -5/x$, $Q = x^5 \cos x$.

pick $C=0$

Step 1: "complement": $\rho = e^{\int P\,dx} = e^{-5\int \frac{1}{x}dx} = e^{-5\ln|x| \cancel{+C}}$.

Also, since we are picking ρ, we can leave out the abs. value, and just use $\rho = e^{-5\ln x} = e^{\ln(x^{-5})} = \boxed{x^{-5}}$.

So we "complement" with ρ: (<u>multiply</u> by ρ)

$$y' - \frac{5}{x} y = x^5 \cos x$$

$$x^{-5} y' - 5x^{-6} y = \cos x$$

$$\left.\begin{aligned} x^{-5} y' + (x^{-5})' y &= \cos x \\ \int (x^{-5} y)' &= \int \cos x \\ x^{-5} y &= \sin x + C \end{aligned}\right\} \text{ step 2, "integrate"}$$

Step 3 "de-complement" (solve for y, by <u>dividing</u> by ρ)

$$\underline{y = x^5 \sin x + C x^5} \quad \text{(general solution)}$$

We can find a <u>particular</u> solution if asked:

given initial value $y\left(\frac{\pi}{2}\right) = 0$

$$\rightarrow \quad 0 = \frac{\pi^5}{32} \sin(\pi/2) + C \frac{\pi^5}{32} = \frac{\pi^5}{32}(1 + C),$$

so $C = -1$, and thus

$$y(x) = x^5 \sin x - x^5 = x^5(\sin(x) - 1)$$

is the solution for the IVP

$$\left\{ y' - \frac{5}{x} y = x^5 \cos x, \quad y\left(\frac{\pi}{2}\right) = 0 \right\}$$

The graph looks like